$\mathbf{Loc} := \mathbf{Frm}^{\mathrm{op}}$

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> Mathematics is not the rigid and rigidity-producing schema that the layman thinks it is; rather, in it we find ourselves at that meeting point of constraint and freedom that is the very essence of human nature.
>
> — Hermann Weyl

Category theory is the study of mathematical analogy, providing a unifying framework for different mathematical domains. As such, it enables valuable knowledge transfer, while also making explicit which properties used in a proof depend on the particular object at play and not on more formal principles. Furthermore, category theoretic proofs are in most part *constructive*, exhibiting an appropriate object as witness of the desired property. In practice, this means that both double negation elimination and choice axioms are avoided.

While point-free topology might also be called *pointless* topology, it is not so. Based on category theoretic language, it provides a bridge between seemingly disconnected subjects: on one hand, we have *logic*, rigid and with discrete steps; on the other, *topology*, rubbery and (by definition) continuous. We are also given a constructive view of topology by focusing on open sets rather than points.

Accordingly, the first part of this paper presents the necessary review of category theory, while the second part is an overview of point-free topology. We give the link between intuitionistic logics and topological spaces, and we characterise the spaces whose points may be recovered entirely from their open sets. We conclude with an informal account of remarkable theorems in topos theory related to pointfree topology, hinting that these spaces may indeed be used for general mathematical reasoning.

1 Categories

Definition 1.1. A *category* consists of *objects* and, for each pair of objects X, Y, *morphisms* $f : X \to Y$, such that:

- morphisms can be composed, i.e. for $f : X \to Y$ and $g : Y \to Z$, there exists $gf : X \to Z$;
- this composition is associative, i.e.

  $$h(gf) = (hg)f$$

  whenever the morphisms h, g and f are composable;
- for each object X, there exists an *identity morphism* 1_X such that for all morphisms $f : X \to Y$, we have

  $$1_Y f = f = f 1_X.$$

Some categories are *concrete*. These have some kind of sets as objects and structure-preserving functions as morphisms. For example, **Grp** has groups as objects and group homomorphisms as morphisms; **Top** has topological spaces as objects and continuous functions as morphisms.

Other categories are *abstract*. These have the elements of a set as objects and some relation as morphisms. The classical examples are posets.

It is often difficult to distinguish particular objects in categories. A more useful notion is that of *isomorphism*: two object $c, d \in \mathsf{C}$ are isomorphic if there are morphisms $f : c \to d$ and $g : d \to c$ such that $fg = \mathrm{id}_d$ and $gf = \mathrm{id}_c$, and we then write $c \simeq d$.

In general, isomorphic objects all share the same categorical properties; conversely, objects with the same categorical properties are isomorphic. Accordingly, in this paper, we will use a generalized "the": when we say that an object is unique, it will always be meant that it is unique up to isomorphism. Consider, for example, an object 0 such that, for every other object c, there is a unique morphism $! : 0 \to c$. We will call 0 an *initial object*.

Proposition 1.2. *Initial objects are unique.*

Proof. Let a, b be initial objects of C. Then there exist uniquely defined morphisms $f : a \to b$ and $g : b \to a$, with compositions $gf : a \to a$ and $fg : b \to b$. But endomorphisms of initial objects must also be unique, and every object must have an identity morphism. Therefore, the compositions are the identities, and a, b are isomorphic. □

From a category C we may form its *opposite* category C^{op} having the same objects as C but with all arrows reversed: whenever C has a morphism $f : c \to d$, C^{op} has a morphism $f^{\mathrm{op}} : d \to c$, and the composition fg in C gives a morphism $g^{\mathrm{op}} f^{\mathrm{op}}$ in C^{op}. The interest of this construction is that every proof of a property for C gives a dual proof for C^{op}. For example, an initial object of C^{op} is a *terminal object* of C, and reversing arrows in the proof above shows that they are also isomorphic. In category theory, we always have two proofs for the price of one.

Example. Let $(P, \leq)$ be a poset with $p, q \in P$, and set an arrow $p \to q$ whenever $p \leq q$. This defines a category, with at most one arrow between any two objects. Moreover, by the antisymmetry of $\leq$, isomorphic objects are equal. The initial element is the infinimum and the terminal element is the supremum, if they exist.

The reader may then ask the natural recursive question: if we were to form the category of categories, what would be the structure-preserving morphisms?

Definition 1.3. For two categories C, D, a *functor* $F : \mathsf{C} \to \mathsf{D}$ consists of

- for each object c of C, and object Fc of D;

- for each morphism $f : c \to d$ of C, a morphism $Ff : Fc \to Fd$ of D.

such that

(1) for all objects c of C, $F(1_c) = 1_{Fc}$;

(2) for all composable morphisms g, f of C, $F(gf) = (Fg)(Ff)$.

This notion is also called a *covariant* functor, in contrast with *contravariant* functors, where in place of requirement (2), we have $Ff : Fb \to Fa$ and $F(gf) = F(f)F(g)$ (i.e. it reverses arrows). Equivalently, a contravariant functor $\mathsf{C} \to \mathsf{D}$ is a covariant functor $\mathsf{C}^{\mathrm{op}} \to \mathsf{D}$.

Note that there are set-theoretic size issues when constructing the category of categories. Is the category of categories itself an object of the category of categories? The usual solution is to work in a given *universe* $\mathcal{U}$ of sets, and we say that a set X is *small* if $X \in \mathcal{U}$, and large otherwise. However, since we will work mostly with small categories, these size matters will not concern us here.

Functors can be composed just like morphisms can. Indeed, from functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{E}$, we have $GF : \mathsf{C} \to \mathsf{E}$ obtained in the obvious manner. As with objects, we say that categories are isomorphic if there are functors $F : \mathsf{C} \to \mathsf{D}$ such that $FG = \mathrm{id}_{\mathsf{D}}$ and $GF = \mathrm{id}_{\mathsf{C}}$.

Note also that we may not form the set of all objects in the concrete categories given above, just as we may not form the set of all sets. There is, however, only a set's worth of arrows between any two objects. Whenever this is the case, we say that a category is *locally small*. We write $\mathsf{C}(a, b)$ for the morphisms between the objects a, b of a locally small category C. We can also "move" between the sets $\mathsf{C}(a, b)$ and $\mathsf{C}(a, c)$ by applying a morphism $f : b \to c$ pointwise. This defines a functor.

Definition 1.4. For C a locally small category and c an object of C, the covariant and contravariant *functors represented by* c are

$$\begin{array}{ccccccc} \mathsf{C} & \xrightarrow{\mathsf{C}(c,-)} & \text{Set} & \quad & \mathsf{C}^{\text{op}} & \xrightarrow{\mathsf{C}(-,c)} & \text{Set} \\ d & \mapsto & \mathsf{C}(c,d) & & d & \mapsto & \mathsf{C}(d,c) \\ \downarrow f & \mapsto & \downarrow f_* & & \downarrow f & \mapsto & f^* \uparrow \\ e & \mapsto & \mathsf{C}(c,e) & & e & \mapsto & \mathsf{C}(e,c) \end{array}$$

where $f_*(g) = fg$ and $f^*(g) = gf$ is post- and pre-composition by f.

A functor $F : \mathsf{C} \to \mathsf{D}$ is *fully faithful* if it induces a bijection

$$\mathsf{C}(x,y) \cong \mathsf{D}(Fx, Fy).$$

for any $x, y \in \mathsf{C}$.

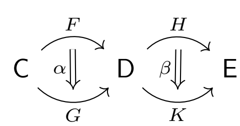
The astute reader will now wonder what would be the morphisms in a category of functors.

Definition 1.5. For $F, G : \mathsf{C} \to \mathsf{D}$ two functors, a *natural transformation* $\alpha : F \Rightarrow G$ is a collection of morphisms $\alpha_c : Fc \to Gc$ in D for each object $c \in \mathsf{C}$, such that

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & Gc \\ \downarrow Ff & & \downarrow Gf \\ Fd & \xrightarrow{\alpha_d} & Gd \end{array}$$

commutes for all $f : c \to d$ in C.

As with functors, natural transformations can be composed in the obvious manner. As in the set of functions, the category of functors from C to D with natural transformations as morphisms is written D^{C} (as a mnemonic, they are the functors *falling* from C to D). Small categories with functors and natural transformations assemble into the protoypical *2-category* **Cat**, which is visualized with the following *globular diagrams*:



where F, G, H, K are functors and α, β are natural transformations.

With natural transformations, we may generalize isomorphisms of categories such that the multiplicity of isomorphic objects ceases to matter. We say that functors are *naturally isomorphic* if there are natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow F$ such that $\alpha\beta = \mathrm{id}_G$ and $\beta\alpha = \mathrm{id}_F$. In simpler terms, this means that all the arrows in the square in definition 1.5 are isomorphisms. Two categories are then *equivalent* if there are functors between them whose compositions are naturally isomorphic to their respective identities.

Theorem 1.6. *A functor $F : \mathsf{C} \to \mathsf{D}$ defines an equivalence of categories if and only if*

- *F is fully faithful;*
- *for all $d \in \mathsf{D}$, there exists $c \in \mathsf{C}$ such that $Fc \simeq d$ (F is* essentially surjective on objects*).*

Proof. See [7], theorem 1.5.9, p. 31. □

A fundamental result of category theory, perhaps *the* most fundamental result, is how locally small categories admit a fully faithful embedding into the category of sets. The first formulation shows a correspondence between the natural transformations between represented functors and other **Set**-valued functors. Coming back to posets, the Yoneda lemma will tell us that an element of a poset is defined by its upper or by its lower bounds.

Theorem 1.7 (Yoneda lemma)**.** *For all $F : \mathsf{C} \to \mathbf{Set}$ where C is locally small, and all objects $c \in \mathsf{C}$, there is a bijection*

$$\mathrm{Nat}(\mathsf{C}(c, -), F) \cong Fc$$

which is natural in both F and c.

Proof. Define the functions $\Phi_c : \mathrm{Nat}(\mathsf{C}(c, -), F) \to Fc$ such that $\Phi_c(\alpha) = \alpha_c(1_c)$ and $\Psi_c : Fc \to \mathrm{Nat}(\mathsf{C}(c, -), F)$ such that $\Psi_c(x)_d(f) = Ff(x)$. Naturality means that diagrams of this sort commute

$$\begin{array}{ccc} \mathrm{Nat}(\mathsf{C}(c,-),F) & \xrightarrow{\Phi_c} & Fc \\ \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle Ff} \\ \mathrm{Nat}(\mathsf{C}(d,-),F) & \xrightarrow{\Phi_d} & Fd \end{array} .$$

To paraphrase Lambek and Scott [5], it is a routine exercise to check that these functions are well defined and are inverses of each other, and that the bijection is natural. For a more detailed proof, see [7], theorem 2.2.4, p. 57. □

Theorem 1.8 (Yoneda embedding)**.** *For $c, d \in \mathsf{C}$ a locally small category, the Yoneda embeddings*

$$\begin{array}{ccc} \mathsf{C} & \xrightarrow{\mathrm{y}} & \mathbf{Set}^{\mathsf{C}^{\mathrm{op}}} \\ c & \mapsto & \mathsf{C}(-,c) \\ \Big\downarrow{\scriptstyle f} & \mapsto & \Big\downarrow{\scriptstyle f_*} \\ d & \mapsto & \mathsf{C}(-,d) \end{array} \qquad \begin{array}{ccc} \mathsf{C}^{\mathrm{op}} & \xrightarrow{\mathrm{y}} & \mathbf{Set}^{\mathsf{C}} \\ c & \mapsto & \mathsf{C}(c,-) \\ \Big\downarrow{\scriptstyle f} & \mapsto & {\scriptstyle f^*}\Big\uparrow \\ d & \mapsto & \mathsf{C}(d,-) \end{array}$$

which assign objects to the functor represented by them, define fully faithful functors, natural in c.

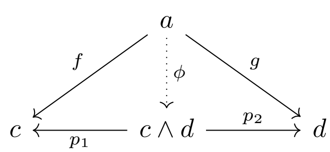
Proof. Apply the Yoneda lemma to $\mathsf{C}(-,d)$. □

What we called the categorical properties of an object is in fact the structure its incoming or outgoing morphism, i.e. the structure of the functor it represents. Then, isomorphic objects obviously define isomorphic represented functors (by the functoriality of the Yoneda embedding). But from this last theorem and the fact that fully faithful functors create isomorphisms, we find that the converse is also true! This justifies our previous assertions regarding the use of a generalized "the".

So far, our study of categories has been mostly descriptive, as we have been working with objects essentially given to us from set-theoretic constructions. However, *limits* (resp. *colimits*) give us a way to construct objects. In general, they can be seen as terminal (resp. initial) objects in a category of diagrams. A formal definition is beyond the scope of this text, as we will mostly work with posets.

The limits that will interest us are products and coproducts, which in posets will end up being the greatest lower bound and least upper bound.

Let c, d be objects in a category C. The *product* of c and d is an object $c \wedge d$ of C along with morphisms $p_1 : c \wedge d \to c$ and $p_2 : c \wedge d \to d$ such that whenever a is an object with morphisms $f : a \to c$, $g : a \to d$, we have a unique $\phi : a \to c \wedge d$ such that the following diagram commutes:



A *coproduct* of c, d, denoted $c \vee d$, is a product in C^{op}. They are the limit and colimit of a discrete diagram of two objects. The reader will confirm that these do in fact define the infimum and supremum of two elements in a poset.

These can be generalized for an arbitrary (small) set $\{c_i\}_{i \in I}$ of objects. For the product, we will now have morphisms

$$p_j : \bigvee_{i \in I} c_i \to c_j$$

such that, if we were to have another object P with morphisms $f_i : P \to c_i$ for $i \in I$, then we have a *unique* $\phi : P \to \bigvee c_i$ such that all the f_i factor as $f_i = p_i \circ \phi$.

In the context of posets, we say $a \wedge b$ is the *meet* of a and b and that $a \vee b$ is their *join*.

Definition 1.9. Two functors $F : \mathsf{C} \rightleftarrows \mathsf{D} : G$ define an *adjunction* between C and D if there is a correspondence

$$\mathsf{D}(Fc, d) \cong \mathsf{C}(c, Gd)$$

for all objects $c \in \mathsf{C}$, $d \in \mathsf{D}$, natural in both entries. We say that F and G are *adjoint functors*, that F is *left adjoint* and G is *right adjoint*.

Adjoint functors are ubiquitous in mathematics and are notable for their limit-preserving properties. However, we will only need them for posets, in which case we have, for $\phi : P \rightleftarrows Q : \psi$ order-preserving maps between posets,

$$\phi(p) \leq q \quad \text{iff} \quad p \leq \psi(q). \tag{1}$$

for all $p \in P$, $q \in Q$.

Proposition 1.10. *Let P, Q be posets, $\phi : P \to Q$ and $\psi : Q \to P$ be order-preserving functions. Then ϕ is left adjoint to ψ if and only if*

$$\phi(\psi(q)) \leq q \quad \text{and} \quad p \leq \psi(\phi(p)) \tag{2}$$

for all $p \in P$ and $q \in Q$.

Proof. Suppose (1). Then, by setting $q = \phi(p)$, we find

$$\phi(p) \leq \phi(p) \quad \text{iff} \quad p \leq \psi \circ \phi(p).$$

Since the left side is a tautology, the right side is always true. It is a similar situation for q, by setting $p = \psi(q)$.

Suppose (2), and let $\phi(p) \leq q$. We have $\psi \circ \phi(p) \leq \psi(q)$ and $x \leq \psi \circ \phi(p)$, whence $p \leq \psi(q)$. It is, again, similar for q. □

Theorem 1.11. *Let $\phi : P \to Q$ be an order preserving map between posets. Then,*

(1) if ϕ has a right adjoint $\psi : Q \to P$, then ϕ preserves arbitrary joins;

(2) if P has all joins and ϕ preserves them, then ϕ has a right adjoint.

Proof. (1) Let $S \subset P$ be such that $\bigvee S$ exists. For $s \in S$, we have $\bigvee S \leq s$, so $\phi\left(\bigvee S\right) \leq \phi(s)$. Now, suppose $\phi(s) \leq b$ for all s, then $s \leq \psi(b)$ for all s as well since ϕ and ψ are adjoint. Therefore, by the property of meets, $\bigvee S \leq \psi(b)$ and again by adjunction, $\phi\left(\bigvee S\right) \leq b$. Therefore, $\bigvee_{s \in S} \phi(s) = \phi\left(\bigvee S\right)$.

(2) We define $\psi(q) = \bigvee\{p \in P | \phi(p) \leq q\}$, which preserves order. Since ϕ preserves joins,

$$\phi(\psi(q)) = \bigvee\{\phi(p) | \phi(p) \leq q\} \leq q$$

and

$$\psi(\phi(p)) \bigvee \{p' | \psi(p') \leq \psi(p)\} \geq p$$

such that by proposition 1.10, ϕ is left adjoint to ψ. $\square$

2 Pointfree topology

2.1 Heyting algebras

In classical mathematical logic, the semantics of a formula is seen through valuations, which assign to each formula the value *true* or *false*, and which distributes naturally over the different connectives: conjunction, disjunction, implication and negation. A formula is then said to be valid if all valuations give it the value *true*, and a formula is provable in classical first order logic if and only if it is valid [1].

In intuitionistic logic, the set $\{\bot, \top\}$ of truth values is replaced by a Heyting algebra, which has all the desirable properties but can be much larger. For example, the open subsets of $\mathbb{R}$ form a Heyting algebra with union and intersection, and it is in fact sufficient to show a formula is valid for $\mathbb{R}$-valued valuations for it to be valid in general; moreover, no finite set of truth values has this property. Like in classical logic, formulae are provable if and only if they are valid [8].

A *lattice* L is a set with distinguished, distinct elements 0 and 1 (symbolizing $\bot$ and $\top$) with *commutative* and *associative* operations

$$\wedge : L \times L \to L, \quad \vee : L \times L \to L,$$

called *meet* and *join* such that the following equations hold for all $x, y, z \in L$:

$$x \wedge x = x, \quad x \vee x = x,$$

$$1 \wedge x = x, \quad 0 \vee x = x,$$

with the *absorbtion laws*

$$x \wedge (x \vee y) = x = x \vee (x \wedge y),$$

A lattice is a *distributive lattice* if it also has the *distributive property*

$$x \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z).$$

These axioms imply the dual statement of distributivity, $x \wedge (y \vee z) = (x \vee y) \wedge (x \vee z)$, by expanding the right-hand side by distributivity twice.

Note that we may recover the *unique* poset strucure of L by setting $x \leq y$ iff $x = x \wedge y$. Indeed, this relation is reflexive by the first equation and transitive by substitution. Suppose $x \leq y \leq z$ in the previous sense. Then, $x = x \wedge y$ and $y = y \wedge z$, whence $x = x \wedge (y \wedge z) = (x \wedge y) \wedge z = x \wedge z$. This fact will allow us to view them as small categories.

As posets, the meet of x and y (resp. join) is their least upper bound (resp. greatest lower bound).

Proposition 2.1. *Let A be a bounded poset with binary products and coproducts. Then A is a lattice in the obvious manner.*

Proof. Only the absorptive laws are non-trivial. Let $z = a \vee (a \wedge b)$. We have that $a \leq z$. But, since $a \leq a$ and $a \leq a \vee b$, universality of z means that $z \leq a$, therefore $a = a$. The dual property follows by duality. □

We will henceforth confuse the two views.

Some examples of lattices are:

- The trivial lattice $2 = \{0, 1\}$.
- The natural numbers with $a \vee b = \text{lcm}(a, b)$ and $a \wedge b = \gcd(a, b)$. The bottom element is 1 and the top element is 0. We can then recover the ordering with $a \leq b$ if and only if $a = \text{lcm}(a, b)$, i.e. a divides b. Then we verify that for all $a \in \mathbb{N}$, a divides 0 and 1 divides a, so they are indeed respectively $\top, \bot$.
- The subsets of a set with $U \leq V$ whenever $U \subseteq V$.
- The open sets of a topological space, again ordered by inclusion.

A lattice is *complete* if it contains all arbitrary meets and joins, denoted

$$\bigwedge_i x_i \quad \text{and} \quad \bigvee_i x_i.$$

A *lattice homomorphism* distributes over the properties above. In particular, it preserves 0 and 1.

Definition 2.2. An *ideal* of a lattice A is a subset $I \subset A$ of A such that

1. $0 \in I$, and for all $a, b \in I$, $a \vee b \in I$
2. $a \in I$ and $b \leq a$ implies $b \in I$.

We say that I is a *prime ideal* if $1 \notin I$ and $a \vee b \in I$ implies $a \in I$ or $b \in I$.

An important example of a prime ideal is the *downward closure* of $a \in A$, the subset $\downarrow(a) = \{b \in A | b \leq a\}$.

Theorem 2.3. *An ideal $I \subset A$ is prime if and only if it is the kernel of a lattice homomorphism $\phi : A \to 2$.*

Proof. Define a function $\phi : A \to 2$ such that $\phi(a) = 0$ if and only if $a \in I$, which is a lattice homomorphism by case analysis.

The converse is trivial. □

In some lattices, we have a operation $x \Rightarrow y$ called *implication*, such that we have the adjunction

$$z \leq (x \Rightarrow y) \quad \text{iff} \quad z \wedge x \leq y. \tag{3}$$

This is justified by the fact it is also the adjunction which defines the universal property of Y^X, the set theoretic space of functions.

There is one more logical operator which we have not yet defined. A proposition is false iff its assertion implies falsity. We therefore define the *negation* of x, $\neg x$, as $x \Rightarrow 0$. From this and (3), we deduce $x \wedge \neg x = 0$.

Definition 2.4. A *Heyting algebra* is a lattice with an implication operation.

Note that Heyting algebras are automatically distributive. Indeed, we have a functor $-\wedge y$ which is a left adjoint, so by theorem 1.11 it preserves coproducts (read: distributes over coproducts), such as $((x \vee y) \wedge z) = (x \wedge z) \vee (y \wedge z)$.

The trivial (read: initial) Heyting algebra is the two element lattice $\{0, 1\}$. These are the classical truth values. Analogously, propositions in classical mathematics are either true or false.

Definition 2.5. A *Boolean algebra* B is a Heyting algebra such that $x \vee \neg x = 1$ holds for all $x \in B$.

There is an analogy between the two operations of a Boolean algebra and those of a Boolean ring. In fact, the category of Boolean algebras is isomorphic to the category of Boolean rings [3].

Proposition 2.6. *In a Boolean agebra B, the following hold:*

(1) $p \Rightarrow q = \neg p \vee q$

(2) $\neg\neg p = p$,

Proof. (1) It suffices to show that for all $x, y, z \in B$, we have $z \leq (\neg x \vee y)$ iff $z \wedge x \leq y$.

(2) We have $p = p \wedge (\neg p \vee \neg\neg p) = (p \wedge \neg p) \vee (p \wedge \neg\neg p) = p \wedge \neg\neg p$, whence $p \leq \neg\neg p$, and the other direction is similar. □

The next insight is that from arbitrary joins, one can define arbitrary meets by taking the join of all upper bounds.

Lemma 2.7. *If P is a poset containing arbitrary joins, then it is complete.*

Proof. Let $S \subset P$. Let $\mathcal{J}$ be the set of upper bounds of S and $L = \bigvee \mathcal{J}$, which straightforwardly satisfies the desired universal property so that $L = \bigwedge P$. The reader may also be convinced of the bijection by applying the Yoneda lemma. □

2.2 Frames and locales

Given a topological space X, its *category of open sets* $\mathcal{O}(X)$, having as objects the open sets of X and a morphism $A \to B$ whenever $A \subset B$. We notice a few things.

A continuous function $f : X \to Y$ induces a map $F : \mathcal{O}(Y) \to \mathcal{O}(X)$, as the continuity of f garantees that there is an open set of X whenever there is an open set B of Y, given by $f^{-1}(B)$. Moreover, this map is a functor, as $A \subset B$ implies that $f^{-1}(A) \subset f^{-1}(B)$.

Conversely, given a functor $F : \mathcal{O}(Y) \to \mathcal{O}(X)$, there will be a corresponding continuous function $f : X \to Y$ though it is not yet clear how to do so, as we would need to recover the points of spaces from their open subsets.

Recall that a *topology* on a set X is a collection of *open sets* $U \subset X$ such that

- both X and $\emptyset$ are open sets;
- for an arbitrary I-indexed collection of open sets U_i, the union $\bigcup_{i \in I} U_i$ is an open set;
- for any pair of open sets U, V (and therefore any finite collection), $U \cap V$ is an open set.

The following definition aims to capture the above formal properties of a topology, with the last one making sure that the objects behave like sets.

Definition 2.8. A *frame* is a poset containing all joins and finite meets which also satisfies the following *infinite distributive law*

$$x \wedge \left(\bigvee_i y_i \right) = \bigvee_i (x \wedge y_i).$$

A homomorphism of frames preserves arbirary joins and finite meets. The frames together with their homomorphisms form the category which is denoted **Frm**.

From the above discussion, we deduce:

Proposition 2.9. *A continuous function $f : X \to Y$ between topological spaces induces a morphism $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ of frames.*

This motivates us to define the geometric analogue of the algebraic frames, quite literally their dual.

Definition 2.10. A *locale* is an object of the category $\mathbf{Loc} := \mathbf{Frm}^{\text{op}}$. The frame corresponding to a locale X is written as $\mathcal{O}(X)$.

Theorem 2.11. *A frame X is*

(1) complete,

(2) a Heyting algebra.

Proof. (1) By lemma 2.7, a frame X has arbitrary joins, so it is complete.

(2) For any $y \in X$, the infinite distributive law says that the functor $y \wedge - : X \to X$ preserves all joins, so it has a right adjoint; call it $- \Rightarrow y : X \to X$. Explicitly, we have

$$z \leq (x \Rightarrow y) \quad \text{if and only if} \quad z \wedge x \leq y$$

which is an implication operation. $\square$

At first glance, this implies that since topologies contain arbitrary unions, they also contain arbitrary intersections! But the meet of open sets was not explicitly defined, so it is not necessarily intersection, simply the infimum relative to the induced Heyting algebra. This in fact implies that *for arbitrary collections $\mathcal{I}$ of open sets, there exists a largest open set included in each $U \in \mathcal{I}$*, which is unobjectionable.

In order words, frames and locales are complete Heyting algebras. However, they are vastly different as categories, as locale homomorphisms need not preserve the implication operation.

To recover the points of a locale, we will use the same observation as in the prologue: a point $x \in X$ of a topological space is a continuous function $x : 1 \to X$, where 1 is the one-point space.

Definition 2.12. A *point* x of a locale L is a morphism $x : 1 \to L$ in **Loc** *or* a morphism $x^{-1} : L \to 1$ in **Frm**, where 1 is the trivial locale $\{0, 1\}$.

We can then construct the set of points of a locale X directly as the set of locale morphisms $p : 1 \to X$, which we write as $\mathrm{pt}(X)$. On an object $U \in X$, the points of U are the subsets $\mathrm{pt}(U) = \{p \in \mathrm{pt}(X) | p^{-1}(U) = 1\}$. These subsets carry a topology. By virtue of the points being frame homomorphisms, we have $p^{-1}\left(\bigvee_i U_i\right) = 1$ if and only if $p^{-1}(U_i) = 1$ for some i, so we have $\mathrm{pt}(\bigvee_i U_i) = \bigcup_i \mathrm{pt}(U_i)$. It is a similar matter for intersections.

Given a locale homomorphism $f : X \to Y$, we can construct its effect on points by composition: for $p : 1 \to X$, we have $fp : 1 \to Y$, and this function is continuous relative to the topology induced by pt.

Definition 2.13. The functor $\mathrm{pt} : \mathbf{Loc} \to \mathbf{Top}$ assigns to a locale its set of points with the induced topology.

Proposition 2.14. *The points of a locale X correspond bijectively to elements $P \in \mathcal{O}(X)$ such that*

$$1 \neq P, \qquad U \wedge V \leq P \quad \textit{iff} \quad U \leq P \textit{ or } V \leq P.$$

Proof. Let $p : 1 \to X$ be a point of X. We know that $\mathcal{I} = \ker(p^{-1})$ is a prime ideal of $\mathcal{O}(X)$ and that this establishes a bijection with points of X. Define $P = \bigvee \mathcal{I}$.

Let $U \in \downarrow(P)$, i.e. $U \leq P$. Then

$$\begin{aligned} p^{-1}(U) &\leq p^{-1}(P) \\ &= p^{-1}\left(\bigvee \mathcal{I}\right) \\ &= \bigvee_{V \in \mathcal{I}} p^{-1}(V) \\ &= \bigvee 0 = 0, \end{aligned}$$

so $U \in \mathcal{I}$.

Conversely, let $V \in \mathcal{I}$. Then, since $P = \bigvee \mathcal{I}$ is the supremum of $\mathcal{I}$, we must have $U \leq P$. The correspondence is then given by

$$p : 1 \to X, \quad \mathcal{I} = \ker(p^{-1}), \quad P = \bigvee \mathcal{I}.$$

$\square$

Theorem 2.15. *The functor* pt : **Loc** $\to$ **Top** *is right adjoint to* Loc : **Top** $\to$ **Loc**.

Proof. Recall that such an adjunction is a natural bijection

$$\mathbf{Loc}(\mathcal{O}(S), X) \simeq \mathbf{Top}(S, \mathrm{pt}(X)).$$

However, since **Frm** is opposite to **Loc**, this becomes

$$\mathbf{Frm}(\mathcal{O}(X), \mathcal{O}(S)) \simeq \mathbf{Top}(S, \mathrm{pt}(X)).$$

Let S be a topological space and X be a locale. The pre-image of a continuous function $f : S \to \mathrm{pt}(X)$ of an open set $\mathrm{pt}(U)$ for $U \in X$ is an open set $f^{-1}(U) \in S$ such that

$$f^{-1}(U) = \{s \in S | f(x)(U) = 1\}.$$

Because f is a continuous function between topological spaces, the induced function $f^{-1} : \mathcal{O}(X) \to \mathcal{O}(S)$ preserves finite meets and arbitrary joins, so it is a frame homomorphism.

Conversely, let $f^{-1} : \mathcal{O}(X) \to \mathcal{O}(S)$ be a morphism of frames. For each $s \in S$, define $g(s) : X \to 1$ such that

$$g(s)(U) = \begin{cases} 1 & \text{if } s \in f^{-1}(U) \\ 0 & \text{otherwise.} \end{cases}$$

This defines a function $g : S \to \mathrm{pt}(X)$, which is clearly continuous as each open set of $\mathrm{pt}(X)$ is given by $\mathrm{pt}(U)$ for some $U \in X$, so that $g^{-1}(U) = f^{-1}(U)$.

To show that this defines a bijection, notice that the construction of g above shows that $g(x)(U) = 1$ if and only if $x \in f^{-1}(U)$, which means that $f(x)(U) = 1$, so $f(s) = g(s)$. Naturality is trivial. $\square$

Geometrically, it is clear that each point x of a space X corresponds to the open set $X - \overline{\{x\}}$, whose downward closure defines a prime ideal. When this correspondence is bijective, X is said to be sober.

Definition 2.16. A topological space S is said to be *sober* when for any open subset $P \in S$ with

- $P \neq S$;
- $U \cap V \subset P$ implies $U \subset P$ or $V \subset P$,

there is a unique point $x \in S$ such that $P = S - \overline{\{x\}}$.

The points of sober spaces can be recovered solely from their open sets. A closed subset is *irreducible* if it cannot be written as the union of two smaller closed sets. Framing it in terms of closed sets, a space is sober if every non-empty irreducible closed set is the closure of a point.

Proposition 2.17. *If S is Hausdorff – i.e. for every distinct points $x, y \in S$ there are open sets $U, V \subset S$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$ – then S is sober.*

Proof. Let F be an non-empty irreducible closed set, and suppose $x, y \in F$ are distinct. Then since S is Hausdorff, there are two non-empty disjoint open sets $U_x \ni x$ and $U_y \ni y$, so $F = (F - U_x) \cup (F - U_y)$ is a decomposition of F into distinct closed sets, so F is not irreducible, which is a contradiction. □

Conversely, some locales do not have "enough points". An extreme example is the locale of surjections $\mathbb{N} \to \mathbb{R}$, which, as a, space does not have any points at all. Indeed, we may define the open sets of this space as surjective partial functions, when are then injective maps from a subframe of $\mathcal{O}(\mathbb{R})$ to $\mathcal{P}(\mathbb{N})$ [2]. For our spaces to make sense as, well, spaces, we will insist on their open sets being defined solely by their points.

Definition 2.18. A locale L has *enough points* if for any $U, V \in \mathcal{O}(L)$,

$$\mathrm{pt}(U) = \mathrm{pt}(V) \quad \text{if and only if} \quad U = V.$$

Equivalently, this means that whenever $U, V \in \mathcal{O}(X)$ are distinct, we can find a point $p : 1 \to X$ such that $p^{-1}(U) \neq p^{-1}(V)$.

Proposition 2.19. *Let X be a locale and S a topological space. Then,*

(1) $pt(X)$ is sober and

(2) $Loc(S)$ has enough points.

Proof. (1) Since every open subset of $\mathrm{pt}(X)$ is of the form $\mathrm{pt}(U)$ for $U \in X$ (and vice-versa), the conditions of sobriety directly translate to the conditions in proposition 2.14, so they are in bijective correspondence with prime ideals $\downarrow(P)$, which are points of X by theorem 2.3. Let $x : 1 \to X$ be the corresponding point, that is, $\ker(x) = \downarrow(P)$. The set $\mathrm{pt}(X) - \overline{\{x\}}$ is equivalently the largest open set not containing x. Suppose we had $V \in X$ such that $U < V$. Since $V \notin \ker(x)$, we have $x \in \mathrm{pt}(V)$, and the bijection above shows that x is unique with this property.

(2) Trivial. □

Theorem 2.20. *The adjunction* $\mathrm{pt} \dashv \mathrm{Loc}$ *restricts to an equivalence of categories between locales X with enough points and spaces S that are sober.*

Proof. It is clear that the unit of the adjunction $\eta : S \to \mathrm{pt}\ \mathrm{Loc}(S)$, i.e. $\mathrm{pt}\ \mathrm{Loc}(\mathrm{id})$, is a bijection sending a point of S to its corresponding point of $\mathrm{pt}\ \mathrm{Loc}(S)$; moreover, η^{-1} sends corresponding open sets to each other, so it is a homeomorphism.

Let $\epsilon : \mathrm{Loc}\ \mathrm{pt}(X) \to X$ be the counit of the adjunction, i.e. $\mathrm{Loc}\ \mathrm{pt}(\mathrm{id})$. The map $\epsilon^{-1} : \mathcal{O}(X) \to \mathcal{O}(\mathrm{pt}(X))$ is surjective, since every open set on $\mathrm{pt}(X)$ is by definition $\mathrm{pt}(U)$ for some $U \in \mathcal{O}(X)$. But the definition of "enough points" shows that this map must also be injective, and ϵ is by definition a locale homomorphism, so it is a locale isomorphism.

By proposition 2.19, the image of pt consists of sober spaces, and the image of Loc consists of locales with enough points. The unit and counit define natural isomorphisms $\mathrm{id} \simeq \mathrm{pt}\ \mathrm{Loc}$ and $\mathrm{Loc}\ \mathrm{pt} \simeq \mathrm{id}$ between sober spaces and locales with enough points, which then defines an equivalence of categories. □

3 Epilogue

> Neurath's ship has become a spaceship.
>
> ---
> Jean-Pierre Marquis, on topos theory.

The study of frames and locales paves the way to topos theory. Informally, a *topos* is a generalization of the category of sets, in that one has

- all finite limits and colimits, such as products, coproducts, initial and terminal objects;
- *exponential objects* representing the functions from X to Y, written Y^X; and
- a *subobject classifier* Ω representing the truth values of the topos.

Any topos is a model of intuitionistic logic. The subobjects of an object in the topos always form a Heyting algebra, in particular those of Ω.

For C a category, the *presheaf* category $\mathbf{Set}^{\mathsf{C}^{\mathrm{op}}}$ is a topos. For example, $\mathbf{Set}^{\mathbb{N}}$ would represent sequences of sets.

Let X be a locale. We may consider its category of presheaves, but we will require that the presheaves respect the inclusions naturally implied by the locale; those presheaves which "glue" together nicely are called *sheaves*, and we write the category of sheaves over X as $\mathrm{Sh}(X)$. Then, the subobject classifier of $\mathrm{Sh}(X)$ has the same structure as X. From the theorems proved in the second chapter, we conclude that

From an arbitrary topological space, we can form a mathematical universe whose truth values are isomorphic to that space.

Furthermore, because the truth values are given as Ω^1, they are also the subsets of the point of the topos. The author does not know what a universe of tori would look like, but it does whet his appetite.

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